A note on the dimensional regularization and the on-mass-shell renormalization in the two-loop order

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Abstract The use of the dimensional regularization in the on-mass-shell renormalization scheme sometimes fails to locally cancel the ultraviolet divergence for a class of diagrams in the two-loop order. The mechanism is discussed based on an example with explicit computation.

1 Introduction

The purpose of this note is to describe a phenomenon which appears in the calculation of Feynman diagrams beyond the one-loop order. Though the content is elementary, its explicit explanation is seldom found in textbooks nor literatures, so that it can be worth to record as a simple memorandum.

The problem discussed in this note is that sometimes the local cancelation of the ultraviolet(UV) divergence seems to be incomplete in the two-loop order.

In this note, a few Feynman diagrams for the self-energy function of a scalar field ϕ are studied as an explicit example to describe the problem mentioned above. While the scalar field ϕ interacts with other fields like the Higgs particle in the standard model, the detailed structure of the interaction is not necessary in the discussion.

The renormalization is performed based on the on-mass-shell scheme.[1)] The ϕ field looks like the Higgs, but no tadpole contribution is considered.

For the regularization of the UV divergence, the dimensional regularization is used where the space-time dimension n is given by

$$n = 4 - 2\varepsilon \ . \tag{1}$$

Here ε is an infinitesimal quantity and the UV divergence is no more abstract infinity but appears as the inverse of ε, so that the infinity is now under control in the computation. The renormalization process is the cancelation of such divergent terms.

In the following sections, the analytic calculation of one- and two-loop diagrams is performed. As it is usually hard to calculate two-loop diagrams exactly, the leading UV singularity is mostly studied. We sometimes call the UV singularity in the form of $\frac{1}{\varepsilon}$ and that of $\frac{1}{\varepsilon^2}$ as the single pole and the double pole, respectively.

2 One-loop two-point function of ϕ and its counter term

The one-loop contribution to the two-point function of ϕ is given by the diagrams in Fig.1. The external momentum q enters in the diagram. In this note, the propagator of ϕ is defined as $1/(p^2 - M^2)$.

$$\Pi_{loop}(q^2) =$$

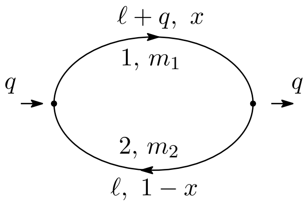

Figure 1: one-loop two-point function of ϕ

$$\Pi_{loop}(q^2) = \int [d\ell] \frac{N}{D_1 D_2}, \quad D_1 = (\ell + q)^2 - m_1^2, \quad D_2 = \ell^2 - m_2^2 \tag{2}$$

where $[d\ell]$ is defined in the appendix-2 and the numerator N is given by

$$N = a + b(q^2) + c(q\ell) + d(\ell^2)\ . \tag{3}$$

Here a, b, c, d are constants which are determined by the particles in the loop and the interaction. The numerator structure is sufficiently general if we follow the standard model with Feynman gauge.

Eq.2 is calculated using formulae given in the appendix-2.

$$\frac{1}{D_1 D_2} = \int_0^1 dx \frac{1}{[xD_1 + (1-x)D_2]^2}. \tag{4}$$

After the momentum shift $\ell \to \ell - xq$ and omitting odd terms in ℓ,

$$\Pi_{loop}(q^2) = \int_0^1 dx \int [d\ell] \frac{N_1}{[\ell^2 - D_{12}]^2} \tag{5}$$

where

$$D_{12} = m_1^2 x + m_2^2(1-x) - q^2 x(1-x), \qquad N_1 = [a + (b - cx + dx^2)q^2] + d\ell^2\ . \tag{6}$$

We perform the ℓ-integration and obtain

$$\Pi_{loop}(q^2) = (FAC_1)\frac{1}{\varepsilon}\int_0^1 dx \left[e + fq^2\right] \frac{1}{D_{12}^{\varepsilon}(q^2)} \tag{7}$$

where $(FAC_1) = \Gamma(1+\varepsilon)(4\pi)^{\varepsilon}/(16\pi^2)$ and

$$e = a + d\frac{2-\varepsilon}{1-\varepsilon}(m_1^2 x + m_2^2(1-x)), \quad f = (b - cx + dx^2) - d\frac{2-\varepsilon}{1-\varepsilon}x(1-x). \tag{8}$$

The x-integrals of e and f are

$$\int_0^1 dxe = \bar{e} + O(\varepsilon) = a + d(m_1^2 + m_2^2) + O(\varepsilon), \qquad \int_0^1 dxf = \bar{f} + O(\varepsilon) = b - \frac{1}{2}c + O(\varepsilon)\ . \tag{9}$$

The derivative of the function is as follows.

$$\Pi'_{loop}(q^2) = (FAC_1)\int_0^1 dx \left[\frac{1}{\varepsilon}f + \frac{(e + fq^2)x(1-x)}{D_{12}(q^2)}\right] \frac{1}{D_{12}^{\varepsilon}(q^2)}\ . \tag{10}$$

The renormalized mass of ϕ-field is denoted by M. From the renormalization condition in the on-mass-shell scheme, the counter term is give by[1)]

$$\Pi_{ct}(q^2) = A(q^2 - M^2) + B, \qquad A = -\Pi'_{loop}(M^2), \quad B = -\Pi_{loop}(M^2) \tag{11}$$

where

$$A = -\frac{1}{\varepsilon}(FAC_1)\bar{f} + O(1), \quad B = -\frac{1}{\varepsilon}(FAC_1)(\bar{e} + \bar{f}M^2) + O(1)\ . \tag{12}$$

Here, A is the wave function renormalization constant and B is the mass renormalization constant. It should be noted that the divergence of constants is determined after fixing $q^2 = M^2$.

Since the renormalization is performed, the following sum is finite at any q^2.

$$\Pi_{loop}(q^2)\ +\ \Pi_{ct}(q^2) =\ \text{finite}\left(\ \text{no}\ \frac{1}{\varepsilon}\ \text{term}\ \right) \tag{13}$$

3 Two-loop two-point function of ϕ

Now we consider the diagrams in Fig.2 and study the sum of them. Naive expectation is as follows: While generally a two-loop diagram can involve the double-pole singularity, the sum shall cancel such terms and the remaining divergence shall be the single-pole type due to the ℓ integration in the figure.

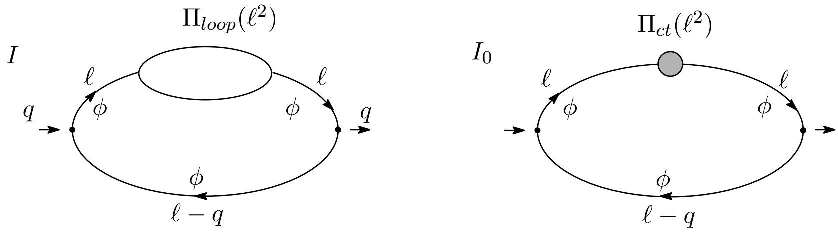

Figure 2: two-loop two-point function of ϕ, I(left) and I_0(right)

First, we calculate I where ℓ_1 and ℓ_2 are the loop momentum of outer large loop and that of inner small loop, respectively, and N is the same in the last section since the numerator of ϕ-propagator is 1.

$$I(q^2) = \int [d\ell_1][d\ell_2]\frac{N}{D_1 D_2 D_3 D_4 D_5}, \tag{14}$$

$$D_1 = (\ell_2 + \ell_1)^2 - m_1^2, \quad D_2 = \ell_2^2 - m_2^2, \quad D_3 = D_4 = \ell_1^2 - M^2, \quad D_5 = (\ell_1 - q)^2 - M^2. \tag{15}$$

In the following, the successive integration is applied to the calculation of I.[2)] The two-loop computation is done by ℓ_2 integral first and ℓ_1 integral second using formulae in the appendix-2. The inner ℓ_2-loop integral is already done in the last section to get Π_{loop} and hereafter we write the outer loop momentum ℓ_1 as ℓ as shown in the figure.

$$I(q^2) = \int [d\ell]\frac{\Pi_{loop}(\ell^2)}{D_3 D_4 D_5}\ . \tag{16}$$

Substituting the expression in the last section, we have

$$I(q^2) = (FAC_1)\frac{1}{\varepsilon}\int_0^1 \frac{dx}{(-x(1-x))^{\varepsilon}}\int [d\ell]\frac{e + f\ell^2}{D_3^2 D_5[\ell^2 - M_X^2]^{\varepsilon}} \tag{17}$$

where D_{12} is written as

$$D_{12}(\ell^2) = -x(1-x)[\ell^2 - M_X^2], \qquad M_X^2 = \frac{m_1^2 x + m_2^2(1-x)}{x(1-x)}. \tag{18}$$

The denominator is combined as

$$\frac{1}{D_3^2 D_5[\ell^2 - M_X^2]^\varepsilon} = \frac{\Gamma(3+\varepsilon)}{\Gamma(1)\Gamma(2)\Gamma(\varepsilon)} \int_{u+v\le 1} dudv \frac{v(1-u-v)^{1-\varepsilon}}{(uD_5 + vD_3 + (1-u-v)[\ell^2 - M_X^2])^{3+\varepsilon}}. \tag{19}$$

After the momentum shift $\ell \to \ell + uq$ and omitting odd terms in ℓ, we have the following ℓ-integral

$$J = \int [d\ell] \frac{h_0 + h_1 \ell^2}{(\ell^2 - D)^{3+\varepsilon}} \tag{20}$$

where

$$D(q^2) = M^2(u+v) + M_X^2(1-u-v) - q^2 u(1-u), \quad h_0 = e + fu^2q^2, \quad h_1 = f\ . \tag{21}$$

The above ℓ-integration is done to obtain

$$J = \frac{(4\pi)^\varepsilon}{16\pi^2}(-1)^\varepsilon \left(-\frac{\Gamma(1+2\varepsilon)}{\Gamma(3+\varepsilon)} h_0 \frac{1}{D^{1+2\varepsilon}} + \frac{(2-\varepsilon)\Gamma(2\varepsilon)}{\Gamma(3+\varepsilon)} h_1 \frac{1}{D^{2\varepsilon}} \right)\ . \tag{22}$$

Finally we have

$$I(q^2) = (FAC_2) \int_0^1 dx \frac{1}{(x(1-x))^\varepsilon} \int_{u+v\le 1} dudv\, v(1-u-v)^{\varepsilon-1} \left(-h_0 \frac{1}{D} + \frac{n}{4\varepsilon} h_1 \right) \frac{1}{D^{2\varepsilon}} \tag{23}$$

where $(FAC_2) = \Gamma(1+2\varepsilon)(4\pi)^{2\varepsilon}/(16\pi^2)^2$.

Next, I_0 is calculated as a one-loop diagram.

$$I_0(q^2) = \int [d\ell] \frac{A(\ell^2 - M^2) + B}{D_3 D_4 D_5} = \int [d\ell] \left(\frac{A}{D_3 D_5} + \frac{B}{D_3^2 D_5} \right)\ . \tag{24}$$

The ℓ-integration leads to

$$I_0(q^2) = (FAC_1) \int_0^1 dx \left[\frac{1}{\varepsilon} A - (1-x) \frac{B}{D_0} \right] \frac{1}{D_0^\varepsilon} \tag{25}$$

where

$$D_0(q^2) = M^2 - q^2 x(1-x)\ . \tag{26}$$

Here the divergence ε^{-1} with A originates from this ℓ-integration.

In the following, we compare $I(M^2)$ and $I_0(M^2)$. The relation $(FAC_2) = (FAC_1)^2(1+O(\varepsilon^2))$ is noted here.

3.1 Single pole

In this subsection, $b = c = d = 0$ is assumed. Then,

$$e = \bar{e} = a, f = \bar{f} = 0, \qquad h_0 = a, h_1 = 0\ . \tag{27}$$

From Eq.23, $I(M^2)$ is

$$I(M^2) = (FAC_2) \int_0^1 dx \frac{1}{(x(1-x))^\varepsilon} \int_{u+v\le 1}^1 dudv\, v(1-u-v)^{\varepsilon-1}(-1)a \frac{1}{D(M^2)} \frac{1}{D^{2\varepsilon}} \tag{28}$$

and we evaluate this after replacing u by $z = 1 - u - v$

$$I(M^2) = (FAC_2)(-a) \int_0^1 dx \frac{1}{(x(1-x))^{\varepsilon}} \int_0^1 dv \int_0^{1-v} dz\, v z^{\varepsilon-1} \frac{1}{D(M^2)} \frac{1}{D^{2\varepsilon}}. \tag{29}$$

We only interested in the contribution of ε^{-1}. Then $D(M^2, z=0) = M^2(1 - v + v^2)$ and

$$I(M^2) \simeq (FAC_2)(-a) \int_0^1 dx \frac{1}{(x(1-x))^{\varepsilon}} \int_0^1 dv \int_0^{1-v} dz\, v z^{\varepsilon-1}$$
$$\left[\frac{1}{D(M^2, z=0)} + \left(\frac{1}{D(M^2)} - \frac{1}{D(M^2, z=0)} \right) \right]. \tag{30}$$

Only the first term has the $O(\varepsilon^{-1})$ contribution which results from z-integration, so that

$$I(M^2) = (FAC_2)(-a) \int_0^1 dv \frac{v}{M^2(1 - v + v^2)} \frac{1}{\varepsilon} + O(1) = (FAC_2)(-a) \frac{1}{\varepsilon} \frac{k_0}{2M^2} + O(1)\ , \tag{31}$$

where

$$k_0 = \int_0^1 \frac{dx}{1 - x + x^2} = \frac{2\sqrt{3}\pi}{9}\ . \tag{32}$$

Next, I_0 is evaluated. Since $\bar{f} = 0, \bar{e} = a$, Eq.25 is

$$I_0(M^2) = (FAC_1) \int_0^1 dx \left[\frac{1}{\varepsilon} A + (1-x)(FAC_1) a \frac{1}{\varepsilon} \frac{1}{D_0(M^2)} \right] + O(1)$$
$$= \frac{1}{\varepsilon} \left[(FAC_1) \int_0^1 dx A + (FAC_1)^2 a \frac{k_0}{2M^2} \right] + O(1). \tag{33}$$

Here A is a finite term.

Result-1

The single-pole divergence cancels between I and B-term contribution of I_0. The sum $I + I_0$ has divergence of $O(\varepsilon^{-1})$ and the divergence comes from the outer ℓ-integral of A-term. Since A-term has the structure of $A(\ell^2 - M^2)$, it cancels one denominator in ℓ-loop to produce divergence.

3.2 Double pole

In this subsection, b, c, d are not vanishing. We only interested in the contribution of $O(\varepsilon^{-2})$.

In Eq.23, $O(\varepsilon^{-2})$ contribution comes from h_1-term.

$$I(M^2) = (FAC_2) \int_0^1 dx \frac{1}{(x(1-x))^{\varepsilon}} \int_{u+v \le 1}^1 du dv\, v(1-u-v)^{\varepsilon-1} \frac{1}{\varepsilon} h_1 + O(\varepsilon^{-1})\ . \tag{34}$$

Here uv-integral gives (as is in the last subsection)

$$\int_{u+v \le 1}^1 du dv\, v(1-u-v)^{\varepsilon-1} = \frac{1}{2\varepsilon} + O(1)\ . \tag{35}$$

Then we have

$$I(M^2) = (FAC_2) \frac{1}{2\varepsilon^2} \bar{f} + O(\varepsilon^{-1})\ . \tag{36}$$

In Eq.25, $O(\varepsilon^{-2})$ contribution comes from A-term

$$I_0(M^2) = (FAC_1)\int_0^1 dx\frac{1}{\varepsilon}A + O(\varepsilon^{-1}) = -(FAC_1)^2\frac{1}{\varepsilon^2}\bar{f} + O(\varepsilon^{-1}) \ . \qquad (37)$$

Result-2

The leading double-pole divergence dose not cancel between I and I_0. The factor 2 difference exists in the result. This is annoying as it differs from the expectation stated in the beginning of this section.

4 Discussion

We have checked the results in the last section as follows. The integral I can be calculated by the standard Nakanishi formula[3,4] and it is given in the appendix-1. The obtained formula is examined analytically and it gives the same leading UV divergence. Complete analytic calculation of I in the last section nor that in the appendix can not be possible. However, numerical treatment is possible to evaluate the value of I. Especially, the direct computation method(DCM)[5,6] is able to compute the coefficients of Laurent expansion of I in ε. The value of the coefficients of $\varepsilon^{-1}(\varepsilon^{-2})$ in the single(double) pole case agrees with analytical results in good numerical accuracy.[7] Thus the analytic evaluation in the last section is confirmed numerically.

Another method to check Eq.36 is to expand the numerator of Eq.14 as

$$N = \left(a + bM^2 + \frac{c}{2}(m_1^2 - m_2^2 - M^2) + dm_2^2\right) + \frac{c}{2}D_1 + \left(d - \frac{c}{2}\right)D_2 + \left(b - \frac{c}{2}\right)D_3 \ . \qquad (38)$$

Then I is described by the linear combination of 4 two-loop integrals whose numerators are 1. Among them only D_3 term gives $O(\varepsilon^{-2})$ contribution,[8] i.e.,

$$\int [d\ell_1][d\ell_2]\frac{1}{D_1D_2D_4D_5} = (FAC_2)\frac{1}{2\varepsilon^2} + O(\varepsilon^{-1}) \ . \qquad (39)$$

The source of discrepancy in the result-2 can be traced through the calculation. In section 2, the renormalization constants A, B are calculated using the value of $D_{12}(q^2)$ at $q^2 = M^2$ and expanded in ε to extract the divergent component assuming that ε is infinitesimal. On the other hand, in the two-loop calculation D_{12} is treated as $D_{12}(\ell^2)$ and in the integral $|\ell^2|$ extends to infinity. In the derivation of Eq.23, $(\ell^2)^\varepsilon$ from Π_{loop} contributes to a factor $\Gamma(2\varepsilon)$ in the integral J. An inadvertent interchange of the two limiting processes, $\varepsilon \to 0$ and $|\ell^2| \to \infty$, seems to result the discrepancy.

For the regularization of UV divergence, we use the dimensional regularization in this note. When we switch to the Pauli-Villars regularization[9] in the double pole case, the coefficients of $(\log\Lambda^2)^2$ cancels each other in the sum of $I + I_0$. Here Λ^2 is the large cutoff parameter in the method. This suggests that the problem presented in this note is related to the dimensional regularization.

A possible modification to avoid the problem is as follows. Though Eq.12 is the standard formula for A, B, the mass-dimension differs from Eq.11, so that a modified counter term can be proposed as

$$\Pi_{ct}(q^2) = [A(q^2 - M^2) + B](M^2 - q^2)^{-\varepsilon} \ . \qquad (40)$$

When we take the limit $\varepsilon \to 0$, this formula is the same as the conventional one. Then we have

$$I_0(q^2) = \int [d\ell]\frac{[A(\ell^2 - M^2) + B](M^2 - q^2)^{-\varepsilon}}{[\ell^2 - M^2]^2[(\ell - q)^2 - M^2]}. \qquad (41)$$

After some manipulation, we obtain the following.

$$I_0(q^2) = (FAC_1)\frac{\Gamma(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)}\int_0^1 dx \left[\frac{1}{2\varepsilon}Ax^\varepsilon - x^{1+\varepsilon}\frac{B}{(1+\varepsilon)D_0}\right]\frac{1}{D_0^{2\varepsilon}} \ . \tag{42}$$

When we use the modified $I_0(M^2)$, the result-1 in the single pole is unchanged while the factor 2 contradiction of the result-2 in the double pole is resolved.

Final and reasonable assessment is that we calculate the following sum when we study the two-loop two-point function

$$I + I_0 + I_{2ct} \tag{43}$$

where I_{2ct} is the counter term in the two-loop order(g^4). This counter term is chosen to make the sum finite, so that the incomplete cancelation in $I + I_0$ is irrelevant.

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Appendix-1

One of the standard method to compute multi-loop integrals is Nakanishi formula.[3,4] In case of two-loop integrals, the process is as follows. The target is the following integral. N is the numerator and K is the number of propagators.

$$I = \int [d\ell_1][d\ell_2]\frac{N}{D_1 D_2 \cdots D_K} \ . \tag{44}$$

First denominator D_j's are combined into a single denominator using Feynman parameters. Then, through the sequence of linear transformation (a momentum shift, a rotation, a scale transformation) of loop momenta ℓ_1, ℓ_2, the denominator turns to $(\ell_1^2+\ell_2^2-V)^K$. The numerator is also transformed in this process. Then simple integration gives

$$I = \int dx_1 \cdots dx_K \delta(1-\sum_{j=1}^{K} x_j)\sum_a C_a \frac{F_a}{U^{n/2}V^{K-n-a}} \tag{45}$$

where

$$C_a = (-1)^{K+a}\frac{\Gamma(K-n-a)}{(4\pi)^n}, \quad V = \sum_{j=1}^{K} m_j^2 x_j - \frac{W}{U} \ . \tag{46}$$

Here function F_a stands for the contribution from a part of numerator which has $2a$ loop momenta. The function $U(W)$ is the sum of 2nd(3rd) order monomials of Feynman parameters.

For I in Fig.2 they are as follows:

$$U = (x_1+x_2)(x_3+x_4+x_5) + x_1 x_2, \qquad W = q^2[x_5((x_1+x_2)(x_3+x_4)+x_1x_2)] \ . \tag{47}$$

The formulae for F_a are omitted here.

Appendix-2

In order to make this note self-contained, basic formulae for one-loop computation are summarized in this appendix.

$$\int [d\ell] \frac{(\ell^2)^\alpha}{(\ell^2 - D)^\beta} = (-1)^{\alpha+\beta} \frac{(4\pi)^\varepsilon}{16\pi^2} \frac{\Gamma(2-\varepsilon+\alpha)\Gamma(\beta-\alpha-2+\varepsilon)}{\Gamma(2-\varepsilon)\Gamma(\beta)} \frac{1}{D^{\beta-\alpha-2+\varepsilon}} \tag{48}$$

where

$$[d\ell] = \frac{d^n\ell}{(2\pi)^n i} \,. \tag{49}$$

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dudv\, \delta(1-u-v) \frac{u^{\alpha-1} v^{\beta-1}}{[Au+Bv]^{\alpha+\beta}} \,. \tag{50}$$

$$\frac{1}{A^\alpha B^\beta C^\gamma} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^1 dudvdw\, \delta(1-u-v-w) \frac{u^{\alpha-1} v^{\beta-1} w^{\gamma-1}}{[Au+Bv+Cw]^{\alpha+\beta+\gamma}} \,. \tag{51}$$

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