

積分ルーチン DICE の性能テスト

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Performance test of the integration routine DICE

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Abstract

自動積分ルーチン DICE を用いてテスト用の典型的な被積分関数と高い特異性を有する被積分関数の定積分を計算した。DICE による自動積分の現状を報告する。

We estimate several definite integrals of the typical testing functions and the highly singular functions by using the automatic integration routine DICE. Present status of the automatic integration with DICE is reported.

Keywords: Automatic Integration, Numerical Integration

1 Introduction

Automatic integration is very important in many fields of science and technology. Here we define the task of automatic integration is to estimate automatically the definite integrals in several variables:

$$I = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \cdots \int_{a_N}^{b_N} dx_N f(x_1, x_2, \dots, x_N). \quad (1)$$

DICE is an automatic integration routine proposed in Research Reports of Kogakuin University[1]. The program package has been studied and developed since 1991 by the present author. The purpose of the routine is to estimate wide range of the definite integrals which are generally given by Eq.(1). The integrand $f(x_1, x_2, \dots, x_N)$ is not only ordinary function but also some kinds of singular function.

DICE adopts an unique division method to divide the integral region into many small subregions to estimate the integral I within the required accuracy. For this purpose the integral region is divided into 2^N subregions. A subregion is divided further into 2^N subregions repeatedly until the convergence conditions are met.

DICE utilizes two sampling methods, regular sampling and random sampling, by which DICE calculates dividing conditions and makes a decision for each subregion to divide the subregion further into 2^N subregions, otherwise proceeds to the estimate step.

In the estimation DICE evaluates the integral estimate \hat{I}_i and the variance estimate $var(\hat{I}_i)$ of subregion i with applying the Monte Carlo method by taking another random sampling to calculate the estimates independently of the convergence conditions.

All values of \hat{I}_i and $var(\hat{I}_i)$ are summed up to obtain finally the integral estimate \hat{I} and the variance estimate for \hat{I} that is $var(\hat{I}) = \hat{\sigma}_I^2$. They are formally written as follows:

$$\hat{I} = \sum_{subregions} \hat{I}_i, \quad (2)$$

$$\hat{\sigma}_I^2 = \sum_{subregions} var(\hat{I}_i). \quad (3)$$

The algorithm of DICE was proposed in 1992. The program package of DICE version 1.3 was developed for super computer with vector processor[2]. This version shows the best performance up to now from the view point that newly introduced dividing conditions effectively suppress useless divisions. Some details of the integration routine are written in Ref.[2].

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The numerical integration is very important and indispensable to theoretical calculation of the scattering cross section in high energy physics. The program package DICE was intended to be one of candidates of integration routine to estimate automatically singular integrals of the squared scattering matrix elements[3] and to estimate the scattering cross section of the high energy radiative e^+e^- reactions, for example, the radiative muon pair production $e^+e^- \rightarrow \mu^+\mu^-\gamma$ and the radiative Bhabha scattering $e^+e^- \rightarrow e^+e^-\gamma$, which are dominant radiative processes in high energy e^+e^- colliding beam experiments.

DICE takes much more sampling to estimate singular integrals than those with no singularity. Dr. Yuasa made efforts on parallelizing the package in order to enable the CPU time much shorter when executing the program on the computer with many processors for parallel computing[4, 5, 6] in KEK¹.

In this paper we calculate the integrals for automatic integration in Ref.[7] and the integrals of singular functions in Ref.[2] with the range of width parameter ε up to 10^{-10} . The range covers smaller values than those used in the reference so that the singularity becomes much higher. In Ref.[1] we did not calculate such a singular integral with the narrower width because the computing power was not enough to get results within a reasonable time. At that time we used Apollo DN10000 workstation in Kogakuin University. In Ref.[2] we used Fujitsu supercomputer system VPP500 in KEK to calculate the cross section of the process $e^+e^- \rightarrow \mu^+\mu^-\gamma$.

2 Performance test

We use DICE version 1.3 for the performance test. The program is executed on the CPU core of computing server Sapp71² in Kogakuin University.

There are three input parameters for execution. They are input error **ERRIN**, maximum number of iteration

ITMAX and number of sample points **NSAMPL**. We set these parameters as follows:

$$\begin{aligned} \text{ERRIN} &= 0.01\% \quad \text{or} \quad 0.001\%, \\ \text{ITMAX} &= 50 \quad \text{and} \quad \text{NSAMPL} = 100. \end{aligned}$$

ERRIN is the required accuracy for the integral estimate \hat{I} . The estimates of \hat{I} and $\hat{\sigma}_I$ are determined by the iterative calculation. Assume k is the first iteration number that the cumulative estimates satisfy the condition:

$$\frac{\hat{\sigma}^{(k)}}{|\hat{I}^{(k)}|} \times 100\% < \text{ERRIN}, \quad (4)$$

where $\hat{I}^{(k)}$ and $\hat{\sigma}^{(k)}$ are k -th cumulative estimates of the integral and the standard deviation, then these estimates are treated as the integral estimate \hat{I} and the error estimate $\hat{\sigma}_I$, respectively.

The parameter **NSAMPL** is the number of sample points in a subregion for random sampling, which number of sample points are randomly selected to calculate the integrand function for the Monte Carlo integration.

2.1 Integrals for automatic integration in several variables

Some integrals in Ref.[7] can be used for good test of the integration routine DICE. We refer these integrals as J_i , $i = 1, 2, 3, 4, 5, 6, 7$ for convenience. Their expressions are as follows:

$$J_1 = \int_V e^{-x_1^2 - x_2^2 - x_3^2 - x_4^2} dV, \quad (5)$$

where V is the region $x_i \geq 0$ and $\sum x_i^2 \leq 1$,

$$J_2 = \int_{-1}^1 \int_{-1}^1 |x^2 + y^2 - 0.25| dx dy, \quad (6)$$

$$J_3 = \int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy, \quad (7)$$

$$J_4 = \int_0^1 \int_0^1 |x - y|^{1/2} dx dy, \quad (8)$$

$$J_5 = \int_0^1 \int_0^1 \int_0^1 \frac{dx dy dz}{3 - \cos \pi x - \cos \pi y - \cos \pi z}, \quad (9)$$

¹The High Energy Accelerator Research Organization

²Fujitsu PRIMERGY RX4770 M3. Sapp71 is one of computing servers in 24 hours operation at the computer room of the Center for Information Science. The computing resource consists of four Intel Xeon E7-8870v4 2.1GHz processors (50M Cache and 20 CPU cores in a processor) and 512GB DDR-2400 memory with QPI(Quick Path Interconnect) speed 50GT/s.

$$J_6 = \int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{3.75 - \cos \pi x - \cos \pi y - \cos \pi z} \quad (10)$$

and

$$J_7 = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 |x^2 + y^2 + z^2 - 0.125| dxdydz. \quad (11)$$

Table 1 shows all results of the integrals J_i . The estimates of DICE for J_1, J_2, J_3, J_4 and J_6 agree with exact ones within the standard deviation of the integral estimates for both cases of input error 0.01% and 0.001%. The difference $|\text{DICEint} - \text{Exact}|$ can be compared to the standard deviation of the integral estimates. The difference for J_5 is comparable with three times of the standard deviation. However the difference for J_7 is more than ten times of the standard deviation.

2.2 Integrals of singular function

In this section we use three integrals I_1, I_2, I_3 in Ref.[2] for the performance test. Their expressions are

$$I_1 = \int_0^1 \int_0^1 \frac{2\varepsilon y}{(x+y-1)^2 + \varepsilon^2} dxdy, \quad (12)$$

$$I_2 = \int_{-1}^1 \int_{-1}^1 \frac{\varepsilon y^2 \theta(1-x^2-y^2)}{(x^2+y^2-a^2)^2 + \varepsilon^2} dxdy \quad (13)$$

and

$$I_3 = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 dxdydz \times \frac{\varepsilon \sqrt{x^2+y^2+z^2} \theta(1-x^2-y^2-z^2)}{(x^2+y^2+z^2-a^2)^2 + \varepsilon^2}, \quad (14)$$

where the θ function in (13) and (14) is given by

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 1 \\ 0 & \text{for } x < 0 \end{cases} \quad (15)$$

and ε and a are constant parameters for integrands.

The parameter ε can be used to control singularity of the integrands. When the value of ε becomes 0 the integrands of I_1, I_2 and I_3 have singularity.

The singularity in the integrand of (12) lies along the diagonal line in x - y plane whose expression is given by

$x + y = 1$. The parameter a expresses the radius of circle in the integrand of (13) and the radius of sphere in (14). When $0 \leq a \leq 1$ both the circle and the sphere are contained in the integral regions. Singularities of the integrands of (13) and (14) appear around the circumference of the circle and around the sphere surface, respectively. Their peak heights are approximately equal to $O(1/\varepsilon)$.

The singular integrals (12), (13) and (14) are good examples to test the performance of DICE. We control both width and height of the singularity in the integrands by changing ε . We use values 1, 10^{-1} , 10^{-2} , ..., 10^{-9} , 10^{-10} for ε . The range limit is much smaller than that used before in Ref.[2].

We control position of the singularity in the integrands of (13) and (14) by changing a . We use three values 0.2, 0.5 and 0.8 for a to see how DICE is able to automatically detect singularity of the integrands in the integral region.

In Table 2 we give the numerical results of DICE for the integral I_1 . Integral estimates are shown with the standard deviations of the estimates in parenthesis. The least significant digits of all numerical results for the integral and the standard deviation are rounded numbers. The second column of the table shows exact values for the integral. For each value of ε we give two results of the integral estimates for the different input errors, $\text{ERRIN} = 0.01\%$ and 0.001% .

Results in Table 2 show good agreements between estimates of DICE and exact values of the integral I_1 within two standard deviations of estimates. The number of sample points used for integral estimates are shown in the 5-th and the 6-th columns of the table. The more ε small the more sample points are required to obtain a result with the same precision. For one order small input error, the calculation requires, roughly ten times of sample points to obtain the precision results.

Fig. 1 shows the division maps of integral region for (a) $\text{ERRIN} = 0.01\%$ and (b) 0.001% whose data has been taken in the sequence of the DICE estimation for the I_1 integral with setting the value of ε to 10^{-6} .

Integral	Exact	DICE integ(std dev)		# of sampling	
		0.01%	0.001%	0.01%	0.001%
J_1	0.16300	0.16298(0.00002)	0.16300(0.00000)	16384000	156692000
J_2	1.86302	1.8631(0.0002)	1.8630(0.0000)	841600	17622000
J_3	1.64493	1.6448(0.0002)	1.6449(0.0000)	870300	24546800
J_4	0.53333	0.53336(0.00005)	0.53333(0.00001)	492800	6472400
J_5	0.50546	0.50537(0.00003)	0.50545(0.00001)	9894900	125528400
J_6	0.30781	0.30783(0.00003)	0.30781(0.00000)	512000	14280100
J_7	7.02618	7.0189(0.0007)	7.0185(0.0001)	2611200	45887200

 Table 1: Results of $ERRIN = 0.01\%$ and 0.001% for J_i , $i = 1, 2, 3, 4, 5, 6, 7$ are obtained by DICE.

ε	I_1 exact	DICE integ(std dev)		# of sampling		CPUtime(sec)	
		0.01%	0.001%	0.01%	0.001%	0.01%	0.001%
1	0.87765	0.87752(0.0008)	0.87764(0.00001)	1.6×10^5	5.5×10^6	0.00	0.13
10^{-1}	2.4807	2.4805(0.0002)	2.4807(0.0000)	3.6×10^6	2.4×10^7	0.09	0.71
10^{-2}	3.0295	3.0293(0.0003)	3.0295(0.0000)	9.0×10^6	2.1×10^8	0.25	5.76
10^{-3}	3.1258	3.1259(0.0003)	3.1257(0.0000)	5.1×10^7	3.5×10^8	1.43	9.22
10^{-4}	3.1396	3.1398(0.0003)	3.1396(0.0000)	4.4×10^7	2.1×10^9	1.30	59.2
10^{-5}	3.1413	3.1412(0.0003)	3.1413(0.0000)	7.8×10^8	2.4×10^9	21.0	64.0
10^{-6}	3.1416	3.1412(0.0003)	3.1415(0.0000)	1.0×10^9	1.4×10^{10}	27.6	382
10^{-7}	3.1416	3.1415(0.0003)	3.1416(0.0000)	7.6×10^9	1.6×10^{10}	225	441
10^{-8}	3.1416	3.1418(0.0003)	3.1416(0.0000)	2.1×10^{10}	1.8×10^{11}	590	5113
10^{-9}	3.1416	3.1413(0.0003)	3.1416(0.0000)	5.4×10^{10}	5.8×10^{11}	1556	16352
10^{-10}	3.1416	3.1410(0.0003)	3.1416(0.0000)	2.8×10^{11}	1.5×10^{12}	7766	44140

 Table 2: Results of $ERRIN = 0.01\%$ and 0.001% for I_1 are obtained by DICE.

Fig. 1 tells us graphically how the number of subregions for input error 0.001% is larger than that for 0.01% to estimate the same integral. The small input error requires much more sampling to obtain the precise result. For both figures (NSAMPL =) 100 sample points are randomly taken to calculate the integrand in order to estimate the integral value and the variance of the integral in each subregion.

The numerical results of DICE for I_2 in Eq.(13) and those for I_3 in Eq.(14) are shown in Table 3 and Table 4, respectively. Integrand functions of these definite integrals have singularity. The singularity of the I_2 integrand take place around the circumference of the circle $x^2 + y^2 = a^2$ and the singularity for the I_3 integrand take place around the sphere surface $x^2 + y^2 + z^2 = a^2$. Three values 0.2, 0.5 and 0.8 for the parameter a are used to see what results can be

obtained by the automatic integration for the different positions of the singularity.

We use several values of the parameter ε in the integrands of I_2 and I_3 , from 1 to 10^{-10} for I_2 and from 1 to 10^{-9} for I_3 . To obtain the precision results in 4 or 5 digits we have two cases of the input error, $ERRIN = 0.01\%$ and 0.001% in both tables.

Some results of the input error 0.001% are missing from the tables in case the parameter ε is set to very small values since the program requires long CPU time for these parameter values and their calculations have not finished yet.

3 Discussions

The last row of numerical results in Table 1 shows us an example of the estimate in disagreement with

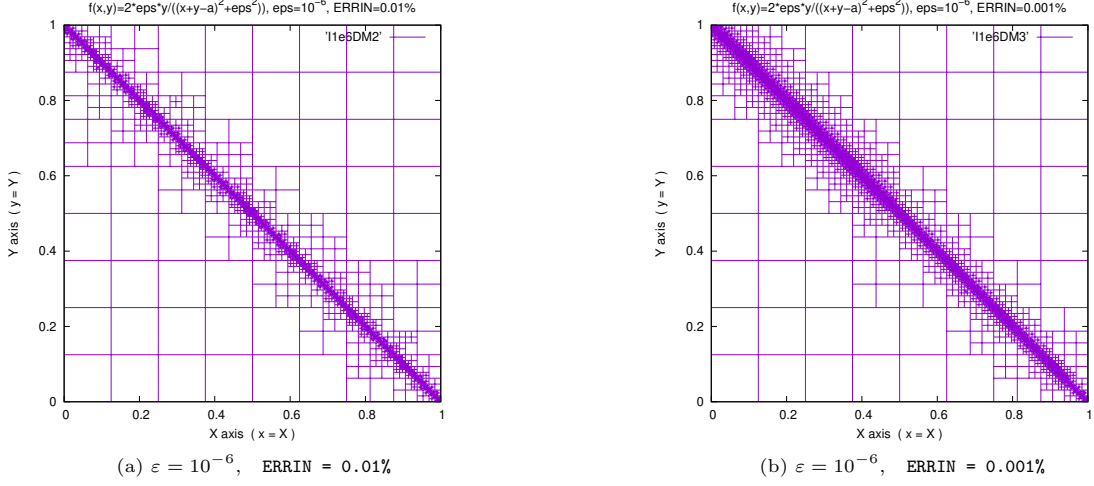


Fig. 1: The 2D plots of division map of the DICE integration for the I_1 integrand $\frac{2\varepsilon y}{(x+y-1)^2 + \varepsilon^2}$ with $\varepsilon = 10^{-6}$ and the input error (a) ERRIN = 0.01% and (b) ERRIN = 0.001%. Data of the figures are automatically generated by DICE to draw them by using GNUPLOT.

the exact value. The difference appears in the third significant digit for both cases of the input error 0.01% and 0.001%. However the standard deviations of the integral estimates give rise only to the difference in the fourth digit in case of the error estimate 0.0007 and only to the deviation in the fifth digit in case of the error estimate 0.0001.

The reason for the disagreement has not yet been understood. As described in the reference[8] the singularity along the sphere of radius 1/2 could affect the result. Similar insufficient results are shown for both Romberg integration and Product midpoint rule in the reference.

Table 2 shows us an excellent performance of DICE. If corners of subregions were not treated as sample points it could not find the corner peaks of the integrand. DICE always treats corners as sample points to hit the corner peaks and to test dividing conditions for subregion. As the result singularity of the I_1 integrand can be detected very well.

In Table 3 the DICE estimates for the I_2 integral are almost good for the ε values from 1 to 10^{-8} for three cases $a = 0.2, 0.5$ and 0.8 except for the case $a = 0.8$ and $\varepsilon = 10^{-8}$ with the input error 0.01%. For very small ε values 10^{-9} and 10^{-10} the integral estimates of DICE are a little bit smaller than the exact ones and the difference $|\text{DICEinteg} - \text{exact}|$ are larger than the estimated error. This tendency is seen more clearly for the outer values of radius a .

We give 3D surface plots of the I_2 integrand in Fig.2. The result of DICE for the integrand given in Fig.2(a) is perfect as shown in the second line of Table 3 since the integrand is not singular. It is well understood from the division map of the integral region for the well behaved integrand as depicted in Fig.3(a).

Fig.2(b) shows the integrand with the most narrow width 10^{-10} . The peak value reaches almost 7×10^9 . For this case the estimate of DICE with the input error 0.01% is shown in the bottom line of Table 3. The agreement of the estimate with the exact value is about 0.1% level. However, the magnitude of the error estimate is one order small. We could not find any reason for the discrepancy from the division map of the result as shown in Fig.3(b).

Fig.2(c) shows the integrand with the radius $a = 1.0$ and $\varepsilon = 10^{-6}$. The estimated value by using DICE for the integral of this function is 2.4647 ± 0.0002 for ERRIN = 0.01% which can be compared to the exact one for the integral 2.4674. The result is obtained by using 1.2×10^{10} sample points and 775 seconds CPU time. Again the difference between the estimate of DICE and the true value is larger than the estimated error by one order of magnitude.

In this case we find several small holes around the circle $x^2 + y^2 = 1$ in Fig.3(c). They are subregions not divided into further smaller ones even though their places are almost around the circumference of the circle where the I_2 integrand has strong singularity as

a	ε	I_2 exact	DICE integ (std dev)		# of sampling		CPUtime(sec)	
			0.01%	0.001%	0.01%	0.001%	0.01%	0.001%
0.2	1	0.56231	0.56227(0.00006)	0.56231(0.00000)	4.1×10^6	8.5×10^7	0.28	5.31
0.2	10^{-1}	0.46055	0.46055(0.00005)	0.46055(0.00000)	3.2×10^6	3.6×10^7	0.22	2.07
0.2	10^{-2}	0.23079	0.23082(0.00002)	0.23079(0.00000)	1.8×10^7	3.2×10^7	1.14	1.07
0.2	10^{-3}	0.20075	0.20073(0.00002)	0.20075(0.00000)	1.9×10^7	1.1×10^8	0.86	3.14
0.2	10^{-4}	0.19773	0.19773(0.00002)	0.19773(0.00000)	9.1×10^7	2.8×10^8	2.74	6.18
0.2	10^{-5}	0.19743	0.19746(0.00002)	0.19743(0.00000)	1.0×10^8	7.9×10^8	2.48	17.2
0.2	10^{-6}	0.19740	0.19739(0.00002)	0.19739(0.00000)	4.6×10^8	1.3×10^9	10.5	29.5
0.2	10^{-7}	0.19739	0.19740(0.00002)	0.19739(0.00000)	1.7×10^9	2.7×10^{10}	39.7	589
0.2	10^{-8}	0.19739	0.19738(0.00002)	0.19739(0.00000)	1.7×10^{10}	2.9×10^{10}	403	622
0.2	10^{-9}	0.19739	0.19732(0.00002)	0.19739(0.00000)	3.0×10^{10}	3.0×10^{11}	716	6740
0.2	10^{-10}	0.19739	0.19735(0.00002)	0.19730(0.00000)	3.6×10^{10}	7.6×10^{12}	882	177211
0.5	1	0.65180	0.65180(0.00006)	0.65181(0.00001)	4.5×10^6	9.7×10^7	0.31	5.87
0.5	10^{-1}	1.1945	1.1944(0.0001)	1.1945(0.0000)	3.1×10^6	5.8×10^7	0.20	3.31
0.5	10^{-2}	1.2300	1.2302(0.0001)	1.2300(0.0000)	6.7×10^6	1.1×10^8	0.36	4.04
0.5	10^{-3}	1.2333	1.2333(0.0001)	1.2333(0.0000)	1.1×10^8	6.9×10^8	5.18	18.1
0.5	10^{-4}	1.2337	1.2337(0.0001)	1.2337(0.0000)	1.1×10^8	1.1×10^9	2.81	24.5
0.5	10^{-5}	1.2337	1.2336(0.0001)	1.2337(0.0000)	5.0×10^8	5.3×10^9	11.8	116
0.5	10^{-6}	1.2337	1.2338(0.0001)	1.2337(0.0000)	1.7×10^9	7.1×10^9	38.6	155
0.5	10^{-7}	1.2337	1.2335(0.0001)	1.2337(0.0000)	5.2×10^9	3.3×10^{10}	124	724
0.5	10^{-8}	1.2337	1.2334(0.0001)	1.2337(0.0000)	1.7×10^{10}	1.2×10^{11}	415	2577
0.5	10^{-9}	1.2337	1.2327(0.0001)	1.2331(0.0000)	5.7×10^{10}	4.2×10^{12}	1409	97635
0.5	10^{-10}	1.2337	1.2311(0.0001)		2.1×10^{11}		5096	
0.8	1	0.74581	0.74586(0.00007)	0.74580(0.00001)	6.2×10^6	1.4×10^8	0.45	8.46
0.8	10^{-1}	2.6436	2.6440(0.0002)	2.6436(0.0000)	2.9×10^6	3.0×10^7	0.19	1.32
0.8	10^{-2}	3.1056	3.1058(0.0003)	3.1056(0.0000)	4.7×10^7	9.5×10^7	3.40	2.51
0.8	10^{-3}	3.1530	3.1532(0.0003)	3.1530(0.0000)	6.6×10^7	4.8×10^8	2.28	11.8
0.8	10^{-4}	3.1577	3.1578(0.0003)	3.1578(0.0000)	2.7×10^8	2.8×10^9	7.05	60.7
0.8	10^{-5}	3.1582	3.1583(0.0003)	3.1582(0.0000)	9.7×10^8	1.3×10^{10}	23.5	284
0.8	10^{-6}	3.1583	3.1578(0.0003)	3.1582(0.0000)	2.5×10^9	1.4×10^{10}	57.5	305
0.8	10^{-7}	3.1583	3.1582(0.0003)	3.1583(0.0000)	9.1×10^9	6.3×10^{10}	214	1368
0.8	10^{-8}	3.1583	3.1574(0.0003)	3.1583(0.0000)	4.6×10^{10}	2.1×10^{11}	1074	4661
0.8	10^{-9}	3.1583	3.1547(0.0003)		2.0×10^{11}		4718	
0.8	10^{-10}	3.1583	3.1552(0.0003)		3.3×10^{11}		7781	

Table 3: Results of the I_2 integral obtained by DICE with the input error $\text{ERRIN} = 0.01\%$ and 0.001% for several combinations of the parameters a and ε .

depicted in Fig.2(c). The reason for some of the holes might be loss of sampling information because all points in the outside of the circle give no information of the integrand except that they are zero due to θ function in the numerator of the integrand.

The 3D surface plots of the integrand function for $a = 0.8, 0.5, 0.2$ and $\varepsilon = 10^{-6}$ are shown in Fig.2(d), (e) and (f). The scale of the Z axis in Fig.2(d) is quite different from that of Fig.2(b) though both 3D surface plots look identical. The results of DICE for the I_2 integral with these values of a and ε are good for both input errors 0.01% and 0.001% as shown in Table 3.

The division maps of the integral region are shown in Fig.3(d), (e) and (f) and all of them are reasonably divided into subregions and even further smaller ones especially around the singular regions.

The estimates of the I_3 integral in Table 4 are almost reasonable up to $\varepsilon = 10^{-6}$. However for the narrower widths $\varepsilon = 10^{-7}, 10^{-8}$ and 10^{-9} the integral estimates are smaller than the exact ones even if taking the error estimates into account. If the error estimates were one digit large, the integral estimate would be consistent with the exact one. One should note that DICE takes much more CPU time to calculate the integral I_3 than I_2 for the same value of the input error. The reason for the longer CPU time required in 3D than in 2D is due to the fact that a subregion is divided into eight subregions in 3D though only into four in 2D.

4 Concluding Remarks

In this paper we have presented the results of DICE v1.3 performance test. The integration routine shows

a	ε	I_3 exact	DICE integ (std dev)		# of sampling		CPUtime(sec)	
			0.01%	0.001%	0.01%	0.001%	0.01%	0.001%
0.2	1	2.24925	2.2488(0.0002)	2.2492(0.0000)	1.0×10^7	1.4×10^8	0.29	4.48
0.2	10^{-1}	1.84220	1.8422(0.0002)	1.8422(0.0000)	1.2×10^7	1.2×10^8	0.42	4.09
0.2	10^{-2}	0.92316	0.92311(0.00009)	0.92316(0.00001)	8.9×10^7	4.0×10^8	2.67	12.1
0.2	10^{-3}	0.80299	0.80305(0.00008)	0.80298(0.00001)	1.2×10^8	1.6×10^9	3.81	53.3
0.2	10^{-4}	0.79091	0.79083(0.00008)	0.79091(0.00001)	4.1×10^8	3.8×10^9	12.9	118
0.2	10^{-5}	0.78970	0.78961(0.00008)	0.78971(0.00001)	3.5×10^9	6.0×10^{10}	111	1865
0.2	10^{-6}	0.78958	0.78961(0.00008)	0.78959(0.00001)	1.9×10^{10}	4.2×10^{11}	770	13673
0.2	10^{-7}	0.78957	0.78931(0.00003)	0.78955(0.00001)	3.3×10^{11}	3.9×10^{12}	11527	125702
0.2	10^{-8}	0.78957	0.78930(0.00008)		4.9×10^{11}		16180	
0.2	10^{-9}	0.78957	0.78896(0.00008)		2.4×10^{12}		76239	
0.5	1	2.6072	2.6068(0.0003)	2.6072(0.0000)	1.1×10^7	1.5×10^8	0.34	4.71
0.5	10^{-1}	4.7781	4.7785(0.0004)	4.7780(0.0000)	6.8×10^6	2.1×10^8	0.20	6.40
0.5	10^{-2}	4.9200	4.9207(0.0004)	4.9201(0.0000)	2.9×10^7	1.3×10^9	0.93	39.5
0.5	10^{-3}	4.9333	4.9337(0.0005)	4.9334(0.0000)	4.5×10^8	4.3×10^9	14.7	134
0.5	10^{-4}	4.9347	4.9346(0.0005)	4.9348(0.0000)	3.3×10^9	6.0×10^{10}	106	2087
0.5	10^{-5}	4.9348	4.9340(0.0005)	4.9349(0.0001)	1.8×10^{10}	3.9×10^{11}	595	13588
0.5	10^{-6}	4.9348	4.9335(0.0005)	4.9348(0.0000)	9.5×10^{10}	1.6×10^{12}	3052	54189
0.5	10^{-7}	4.9348	4.9333(0.0005)		3.4×10^{11}		11467	
0.5	10^{-8}	4.9348	4.9316(0.0005)		1.5×10^{12}		49505	
0.8	1	2.9832	2.9837(0.0003)	2.9832(0.0000)	5.8×10^6	5.5×10^7	0.16	2.27
0.8	10^{-1}	10.575	10.575(0.001)	10.574(0.000)	1.1×10^7	3.6×10^8	0.32	13.8
0.8	10^{-2}	12.422	12.422(0.001)	12.423(0.000)	8.5×10^7	3.2×10^9	2.61	102
0.8	10^{-3}	12.612	12.611(0.001)	12.612(0.000)	8.2×10^8	1.0×10^{10}	25.3	386
0.8	10^{-4}	12.631	12.633(0.001)	12.631(0.000)	5.6×10^9	1.1×10^{11}	174	3662
0.8	10^{-5}	12.633	12.632(0.001)	12.633(0.000)	4.1×10^{10}	7.0×10^{11}	1310	22959
0.8	10^{-6}	12.633	12.632(0.001)		4.0×10^{11}		14488	
0.8	10^{-7}	12.633	12.630(0.001)		1.1×10^{12}		37130	
0.8	10^{-8}	12.633	12.594(0.001)		7.7×10^{12}		248717	
0.8	10^{-9}	12.633	12.626(0.001)		1.2×10^{13}		642699	

Table 4: Results of the I_3 integral obtained by DICE with the input error $\text{ERRIN} = 0.01\%$ and 0.001% for several combinations of the parameters a and ε .

good performance for the test integrals.

However the program does not always produce proper estimate of the integral within the error estimate for parameters of the integral whose integrand has sharp singularity, especially with high peak and very narrow width. For such very singular integrand the program gives small estimate of the integral with small error estimate and the results are incorrect. The reason for the insufficient results of these singular integrals is not understood very well.

Possible reason would be due to poor sampling around the singular regions. We had insufficient results for the integral with the singularity of very narrow width. One can imagine the number of sample points is too limited to hit points near the singularity peak because of narrow width of the singularity. In such a case both the small integral value and the small variance of the integral in several subregions with utilizing the poor sample points would affect the dividing conditions for them such that they are not divided further.

As the results unreasonably small estimates of the integral and the variance in non-negligible number of subregions would be main source of underestimate for both the integral value and the error.

5 Acknowledgement

The author expresses his sincere gratitude to Prof. Ken-ichi Baba, director of the Center for Information Science, for opportunity to use the Sapp71 computing server. The author expresses sincere gratitude to people who maintain the computer in Kogakuin University.

All the figures in the paper are drawn as follows: a small FORTRAN subroutine callable from the main program of DICE integration routine is made to produce data of the division map of integral region and another source code is made to produce 3D data of singular functions. Gnuplot version 5.2[9] analyze these data to draw the figures.

It would be impossible to explain the difficulty of automatic integration to the reader without showing the figures produced by gnuplot. The author expresses sincere gratitude to people who made efforts to develop such a useful graphic analyzer and drawer for public use.

References

- [1] K.Tobimatsu and S.Kawabata, *A New Algorithm for Numerical Integration*. Research Reports of Kogakuin Univ. No.72 (1992) 297-306.
- [2] K.Tobimatsu and S.Kawabata, *Multi-dimensional Integration Routine DICE*. Research Reports of Kogakuin Univ. No.85 (1998) 7-27.
- [3] D.Perret-Gallix, et.al., *Computational Particle Physics: Experimental Needs*. KEK Proceedings 99-23, Feb. (2000) 110-123.
- [4] F.Yuasa, K.Tobimatsu and S.Kawabata, *Experiences on Multi-dimensional Integration Package*. KEK Proceedings 2002-11, Aug. (2002) 120-127.
- [5] F.Yuasa, K.Tobimatsu and S.Kawabata, *Parallelization of the multidimensional integration package: DICE*. Nuclear Instruments and Methods in Physics Research A 502 (2003) 599-601.
- [6] F.Yuasa, K.Tobimatsu and S.Kawabata, *Recent developments in parallelization of the multidimensional integration package DICE*. Nuclear Instruments and Methods in Physics Research A 559 (2006) 306-309.
- [7] P.J.Davis and P.Rabinowitz, *Methods of Numerical Integration*, The Second Edition, ACADEMIC PRESS, 1984.
- [8] See page 455 in the reference [7].
- [9] T.Williams and C.Kelley, gnuplot 5.2, Manual Ver.5.2.5 (October 2018) by D.Crawford. Retrieved from <http://www.gnuplot.info/>

積分ルーチン DICE の性能テスト

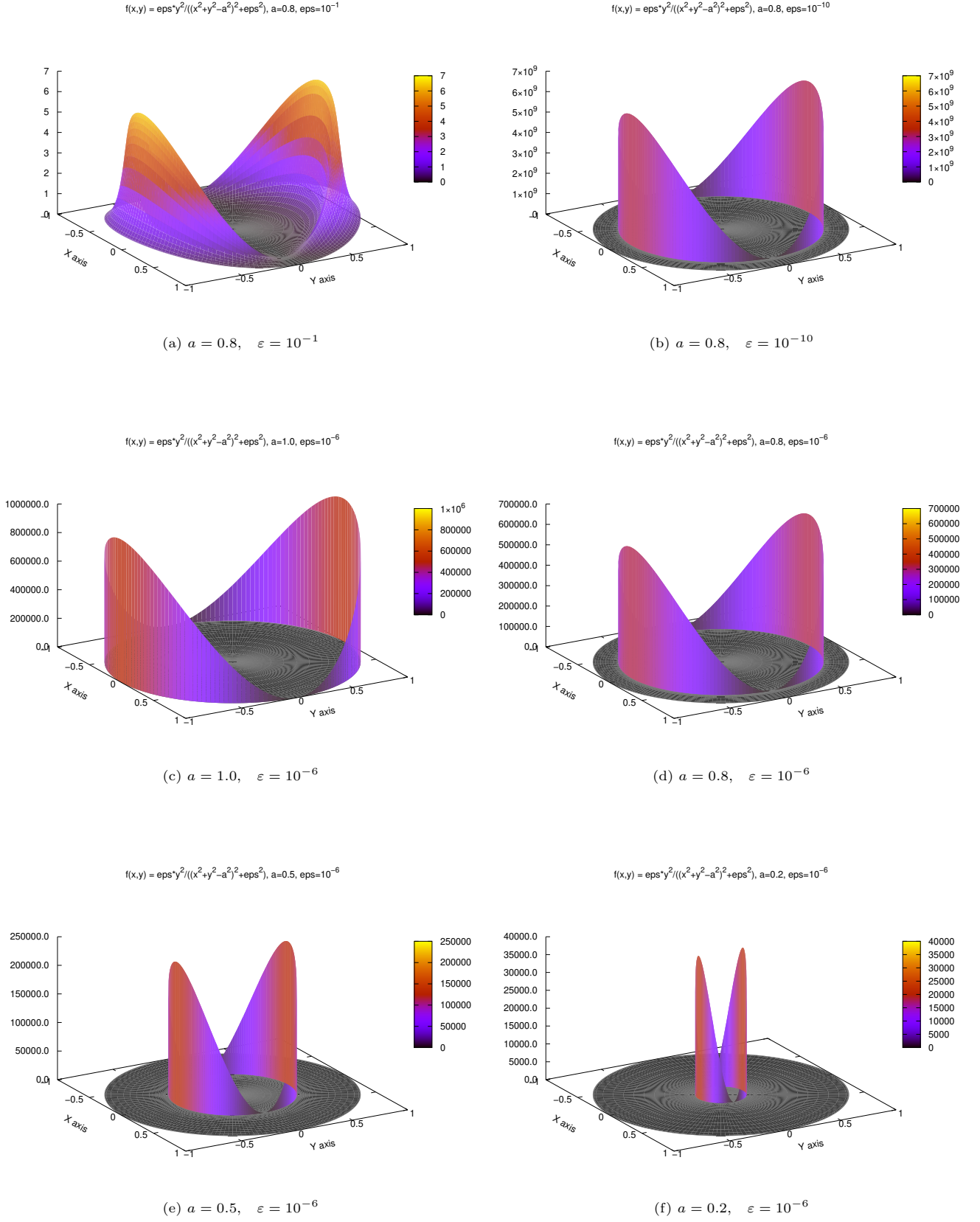


Fig. 2: 3D plots of the I_2 integrand, $\frac{\epsilon y^2 \theta(1 - x^2 - y^2)}{(x^2 + y^2 - a^2)^2 + \epsilon^2}$ with (a) $a = 0.8$ and $\epsilon = 10^{-1}$, (b) $a = 0.8$ and $\epsilon = 10^{-10}$, (c) $a = 1.0$ and $\epsilon = 10^{-6}$, (d) $a = 0.8$ and $\epsilon = 10^{-6}$, (e) $a = 0.5$ and $\epsilon = 10^{-6}$, (f) $a = 0.2$ and $\epsilon = 10^{-6}$. The 3D data of the figures are generated by the author made program to draw them by using GNUPLOT with the pm3d mode.

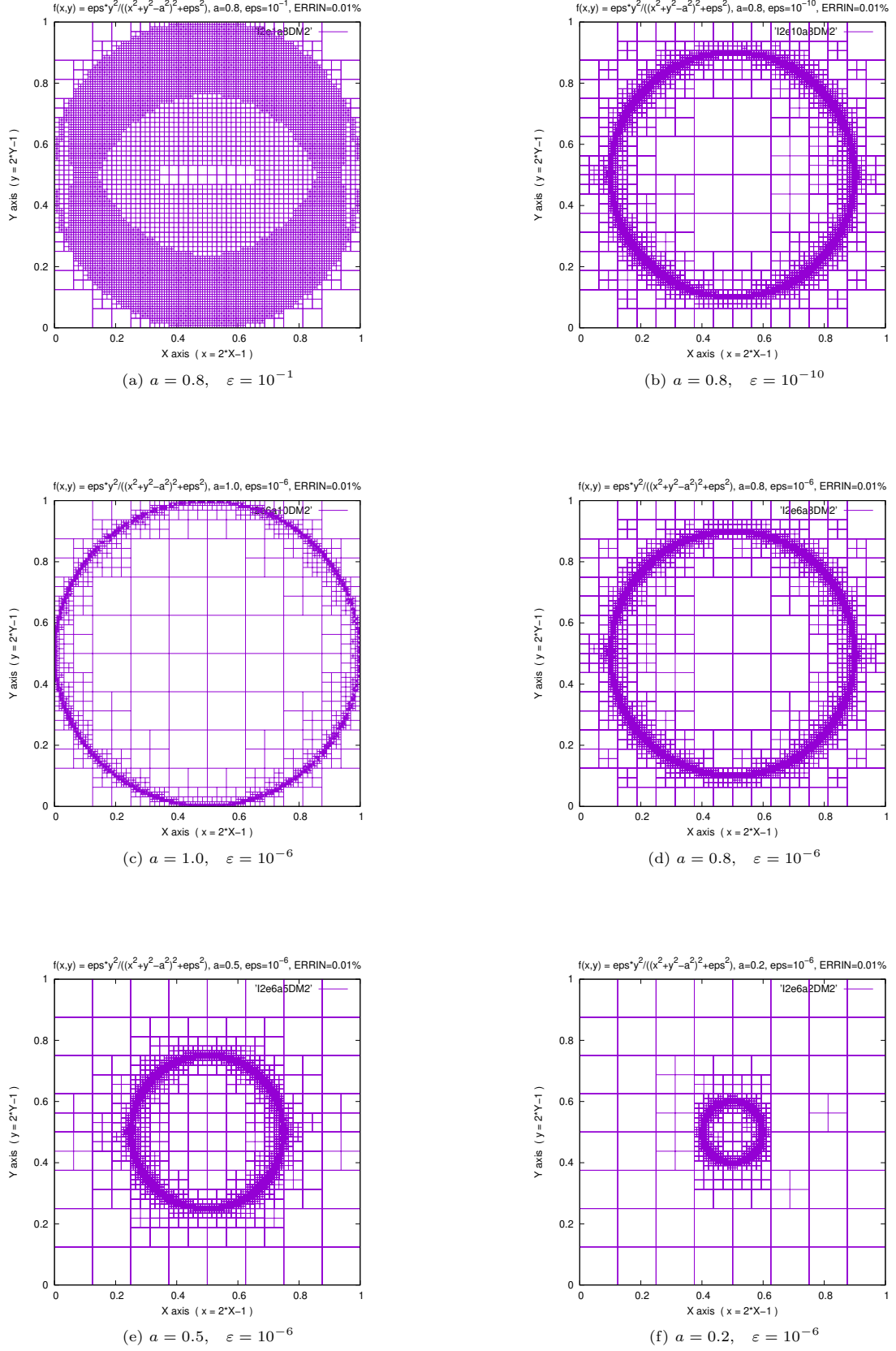


Fig. 3: The 2D plots of division map for the I_2 integrand $\frac{\epsilon y^2 \theta(1-x^2-y^2)}{(x^2+y^2-a^2)^2+\epsilon^2}$, (a) $a = 0.8, \epsilon = 10^{-1}, \text{ERRIN} = 0.01\%$, (b) $a = 0.8, \epsilon = 10^{-10}, \text{ERRIN} = 0.01\%$, (c) $a = 1.0, \epsilon = 10^{-6}, \text{ERRIN} = 0.01\%$, (d) $a = 0.8, \epsilon = 10^{-6}, \text{ERRIN} = 0.01\%$, (e) $a = 0.5, \epsilon = 10^{-6}, \text{ERRIN} = 0.01\%$, (f) $a = 0.2, \epsilon = 10^{-6}, \text{ERRIN} = 0.01\%$, which data are automatically generated by DICE to draw them by using GNUPLOT.